Non-commutative eigenequations and physical applications

We call non-commutative eigenequation an equation of the following form :

$$\sigma_i \otimes X^i |\psi_a(x)\rangle\rangle = (\sigma_i x^i + \lambda_a(x))|\psi_a(x)\rangle\rangle$$

where $(\sigma_i \in \mathcal{L}(\mathbb{C}^2))$ are the Pauli matrices, $(X^i \in \mathcal{L}(\mathbb{C}^N))$ are some three operators, $|\psi\rangle\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^N$ is the unkown eigenvector and $\sigma_i x^i + \lambda_a(x)\mathrm{id} \in \mathcal{L}(\mathbb{C}^2)$ is the unkown non-commutative eigenvalue (it is called non-commutative because it is a matrix) with the condition $\lambda_0(x) = 0$. It is then a generalization of the usual eigenvalue equation. This non-commutative eigenequation can be viewed as the time-independent Schrödinger equation not for a point particle system but for a quantum geometric object (a non-commutative manifold). In this viewpoint, (X^i) are the coordinates observables onto the NC manifold which do not commute ($[X^i, X^j] \neq 0$). In the eigenstate $|\psi_a\rangle\rangle$, the NC eigenvalue of the NC manifold can be characterized by $M_a = \{x \in \mathbb{R}^3, s.t.\ \det(\sigma_i \otimes (X^i - x^i) - \lambda_a(x)) = 0\}$ which is a surface in $\mathbb{R}^3$. NC eigenvalues of a quantum geometric object can be then assimilated to classical manifolds.
NC eigenequations have two main physical applications. In quantum information theory, they describe pseudo steady states $\rho_a = \mathrm{tr}_{\mathbb{C}^N} |\psi_a\rangle\rangle\langle\langle\psi_a|$ of a qubit entangled with an environment described by (X^i). In quantum gravity theory, they describe a quantization of the spacetime, the gravity interaction emerging from this quantization. In particular, the NC eigenequation is associated with the Dirac equation (relativistic quantum equation for a $\frac{1}{2}$-spin particle) in the matrix model of the string theory.

Unfortunately, we know how to solve the NC equation for only two cases : the NC plane ($X^1 + \imath X^2 = a$, $X^1 - \imath X^2 = a^+$, $X^3 = 0$, a,a^+ being annihilation and creation operators), and the NC sphere ($X^i = J^i$ angular momentum operators). The first goal of this project is to study the method used to solve the equation in these two cases and to see problems occuring for two others, the NC ellipsoid and the NC hyperboloid. The second goal of this project is to code a program in order to numerically solve the equation and to draw the graphs of the classical "eigenmanifolds".

mathematical physics, quantum theory, quantum gravity

A very good level in mathematics is required !

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